

# Topology, Random Matrix Theory and the spectrum of the Wilson Dirac operator

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We study the spectrum of the hermitian Wilson Dirac operator in the  $\varepsilon$ -regime of QCD in the quenched approximation and compare it to predictions from Wilson Random Matrix Theory. Using the distributions of single eigenvalues in the microscopic limit and for specific topological charge sectors, we examine the possibility of extracting estimates of the low energy constants which parametrise the lattice artefacts in Wilson chiral perturbation theory. The topological charge of the field configurations is obtained from a field theoretical definition as well as from the flow of eigenvalues of the hermitian Wilson Dirac operator, and we determine the extent to which the two are correlated.

*XXIX International Symposium on Lattice Field Theory*  
*July 10 – 16, 2011*  
*Squaw Valley, Lake Tahoe, California*

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## 1. Introduction

It has been known for some time that the low lying eigenvalues of the QCD Dirac operator are reproduced in the microscopic limit by the eigenvalues of large random matrices with the correct anti-hermitian symmetry structure [1, 2]. The connection between random matrix theory (RMT) and chiral perturbation theory ( $\chi$ PT) has been established within the so-called  $\varepsilon$ -regime of QCD, where pion masses are small enough that their wavelength becomes of the order of the finite size of the lattice. If this condition is enforced, finite volume effects are noticeable even when the volume is taken to infinity. Pion momenta are suppressed and, to leading order, pion fields are constant over the whole lattice while the effect of spontaneous chiral symmetry breaking is captured in a single low energy constant (LEC), the quark condensate  $\Sigma$ . From the Banks-Casher relation it is known that  $\Sigma$  can be defined through the spectral density of the Dirac operator at the origin, so this provides the connection between the spectrum of the Dirac operator and the one from RMT in the microscopic regime.

So far, attempts to obtain physical parameters from numerical simulations of QCD in the  $\varepsilon$ -regime have been restricted to the use of lattice Dirac operators with an exact chiral symmetry [3, 4, 5]. Recent progress has shown, however, that a sensible analysis can also be performed using the standard Wilson Dirac operator, for which the chiral symmetry is explicitly broken. This is made possible in the framework of Wilson  $\chi$ PT that includes the effects of the lattice discretisation and that has recently been formulated in the  $\varepsilon$ -regime [6, 7]. The Lagrangian of this effective low-energy theory,

$$\mathcal{L}(U) = \frac{1}{2}m\Sigma\text{Tr}(U + U^\dagger) - a^2W_6\text{Tr}(U + U^\dagger)^2 - a^2W_7\text{Tr}(U - U^\dagger)^2 - a^2W_8\text{Tr}(U^2 + U^{\dagger 2}),$$

contains on top of the continuum Lagrangian three additional operators which parametrise the lattice artefacts to leading order in  $a$  using the LECs  $W_{6,7,8}$ . Chiral RMT, too, can be extended to include the effects of these additional operators, leading to what is known as Wilson RMT [8, 9]. There is a one-to-one correspondence between the new parameters in both frameworks, so we will use the Wilson  $\chi$ PT nomenclature for both.

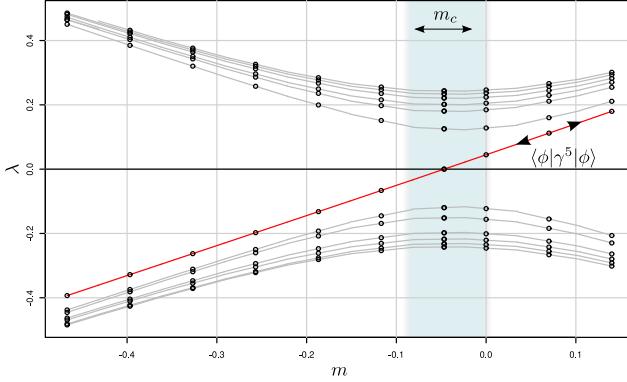
Simulations of QCD, when taken towards the chiral limit at fixed lattice spacing, are likely to run into numerical instabilities due to the onset of  $\varepsilon$ -regime dynamics. To avoid such problems, one needs a thorough understanding of the complicated way the spectrum depends on the different scales like the quark masses, the lattice volume and the lattice spacing. In this context, analytic results concerning the spectral density of the Wilson Dirac operator obtained in [8, 9, 10] have already provided important insights. These proceedings report on a feasibility study, comparing the Wilson RMT description to results from quenched QCD simulations in the  $\varepsilon$ -regime. Our primary interest is whether the effective description can indeed reproduce the important features of the Wilson Dirac spectrum that we measure, and we show that this seems to be the case. A secondary goal is to determine the extent to which one can extract information on the LECs from this setup. Eventually, the results could also have an impact on (Wilson)  $\chi$ PT analysis of dynamical simulations outside of the  $\varepsilon$ -regime, since the same LECs occur in the regular regimes of  $\chi$ PT.

## 2. Definitions of topological charge

The QCD partition function in the  $\varepsilon$ -regime naturally separates into contributions from different topological charge sectors [11]. When constructing Wilson  $\chi$ PT in the  $\varepsilon$ -regime, the presence of the different sectors can be parametrised through a phase factor in the partition function,

$$Z = \sum_v \int_U dU \det[U]^v \exp(V \mathcal{L}(U)),$$

decomposing it in a manner akin to a Fourier transform [8]. As a consequence of this construction the study of the low lying eigenvalues in the Dirac operator spectrum can be done separately for each of the charge sectors, but this obviously requires good control over the topological properties of the system. However, at finite lattice spacing the topological charge can not be defined uniquely and this creates an ambiguity in the assignment of configurations to the different charge sectors. The straightforward definition from the field strength tensor  $v_{FT} \sim \int_V F_{\mu\nu} \tilde{F}^{\mu\nu}$  is attractive, because it can easily be calculated even on large lattices, but for the current purpose, the natural definition of the charge is through the chiral overlap index. One way to calculate it is provided by the eigenvalue flow method [12, 13] which counts the number of real modes of  $D_W$  weighted with the signs of their chiralities. To be specific, the procedure determines those values of the mass  $m$  for which the hermitian operator  $D_5(m)$  satisfies



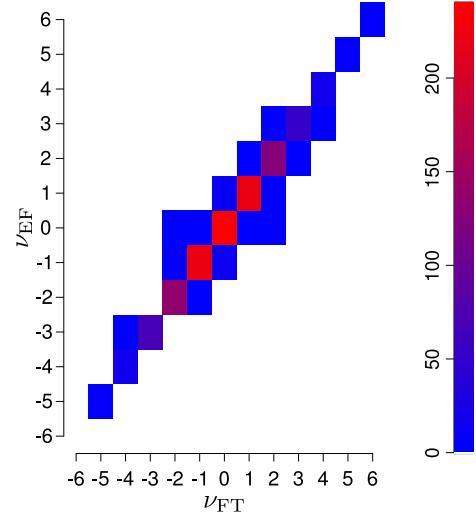
**Figure 1:** Sample of a typical eigenvalue flow calculation for a volume of  $24^3 \times 24$  at a gauge coupling of  $\beta = 6.2$  using thirty levels of HYP smearing. The region around the critical mass  $m_c$  where zeros are expected to occur is indicated in blue, while the eigenmode drawn in red shows a crossing eigenvalue.

$$D_5(m)\psi = \gamma_5(D_W + m)\psi = 0,$$

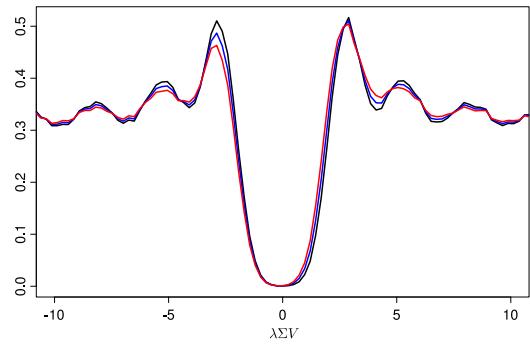
hence the zero modes of  $D_5(m)$  correspond to the real eigenvalues of  $D_W$ . Moreover, perturbation theory shows that the slope of an eigenvalue as a function of  $m$  is given by the chirality of the eigenmode, hence the flow of each eigenvalue can be traced as a smooth function of the mass, as illustrated in figure 1, and the net number of crossings then yields the chiral index. In practice, one needs to choose a cut-off  $m_{cut}$  beyond which no further physically relevant zeros are assumed to exist. Obviously, this has an influence on the value of the index. Sufficiently close to the continuum limit, however, any choice of  $m_{cut} < m_c \simeq 0$  will produce the same value for the index. The approach to this limit can in fact be improved by HYP smearing, and for our current exploratory studies we rather aggressively use thirty levels of HYP smearing.

A comparison between the field theoretic definition  $v_{FT}$  and the one from the eigenflow  $v_{EF}$  is shown in figure 2 for configurations generated using the Wilson gauge action at  $\beta = 6.2$  on a volume of  $20^3 \times 20$ . Both definitions agree for 96.4% of the configurations, while the deviations are almost always limited to  $\Delta v = 1$ . The table included in figure 3 gives estimates for the fraction of deviating results over a range of parameters. For relatively fine lattices and moderate volumes, the field theoretical charge definition and the overlap index agree quite well. The agreement deteriorates as we go to larger volumes, but the effect is mild and justifies the use of the field theoretic definition on lattices for which explicit eigenvalue flow calculations are too expensive. Of course this comes at the cost of introducing a systematic error due to the occasional misinterpretation of the charge. To estimate the size of that error, one can mimic the effect of misassigning charges by mixing measurements from two separate RMT calculations at identical parameters but in different charge sectors. The results of such an exercise are included in figure 3. While the impact of mixing the sectors is discernible, the impact is small enough to have limited impact on the precision of our current preliminary fits, even at large mixing ratios of up to 0.2.

| Parameters                    | $\text{frac}_{v_{FT} \neq v_{EF}}$ |
|-------------------------------|------------------------------------|
| $\beta = 5.9, 14^3 \times 16$ | 0.065                              |
| $\beta = 6.2, 14^3 \times 16$ | 0.008                              |
| $\beta = 6.2, 20^3 \times 20$ | 0.036                              |
| $\beta = 6.2, 24^3 \times 24$ | 0.096                              |



**Figure 2:** A 2D histogram showing the distribution of the topological charge as defined through a field theoretical definition versus the charge defined through the eigenvalue flow on a  $20^3 \times 20$  lattice at  $\beta = 6.2$ .

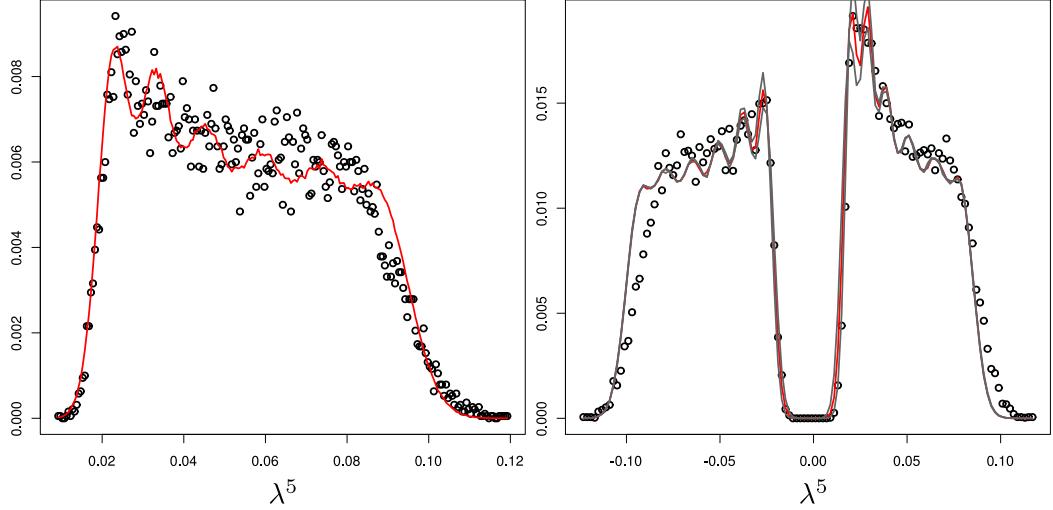


**Figure 3:** *Left:* Table of the estimated fraction of configurations that produce a different result for the topological charge according to the field theoretical and eigenvalue flow definitions. *Right:* RMT simulations of the spectral density of the Wilson Dirac operator at  $v = 0$ , displaying the effect of mixing in 0% (black), 10% (blue) and 20% (red) configurations with charge  $v = 1$ .

### 3. Wilson RMT and the Wilson Dirac operator spectrum of quenched QCD

With these preliminaries handled, we turn to the fitting of our lattice data. One prerequisite for a sensible comparison is the availability of eigenvalues small enough to be within the  $\varepsilon$ -regime. When simulating at a Wilson gauge coupling of  $\beta = 6.2$ , the smallest volume producing a reasonable number of eigenvalues without apparent bulk effects, but with a mass gap, turned out to be

$24^3 \times 24$ . At this volume the fraction of misassigned configurations is a manageable ten percent, so we use the field theoretical definition of the topological charge instead of the Wilson flow charge.

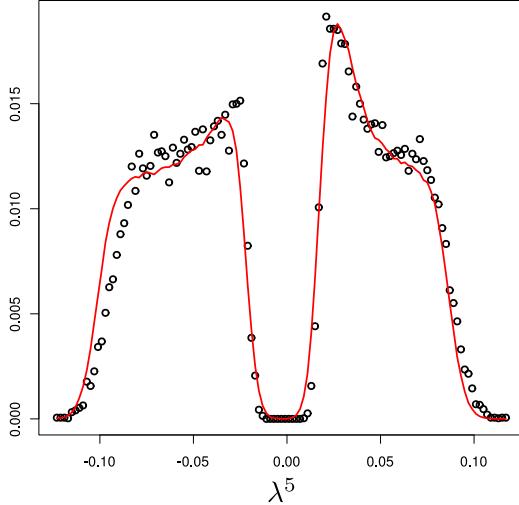


**Figure 4:** Distributions of the 12 lowest lying eigenvalues of the hermitian Wilson Dirac operator in the sectors  $v = 0$  (left) and  $v = 1$  (right) together with fits including the effects of the operator  $W_8$ . Results are for a volume of  $24^3 \times 24$  at  $\beta = 6.2$ . The gray lines in the right panel indicate the effects of varying the value of  $W_8$  by  $\pm 10\%$ .

In figure 4 we show two samples of the distribution of the 12 lowest lying eigenvalues of the Wilson Dirac operator in the charge sectors  $v = 0$  and  $1$  for a volume of  $24^3 \times 24$  at  $\beta = 6.2$ . The spectrum does not appear to exhibit the undulating pattern that can, for example, be seen in the RMT data of figure 3. If this was down to statistics only, we would require an increase in statistics by an order of magnitude. However, rather than using the formulae of [9] for the full spectrum, RMT can be used to extract distributions for the single eigenvalues. This provides additional information which can be used in the fitting procedure [14].

We therefore implemented a Monte Carlo spectrum calculation using Wilson RMT. The spectra were then used to fit the histograms of the separate eigenvalues, the results of which are also shown in figure 4. To estimate the precision that can be reached in the determination of the LECs by such an approach, we vary the value of  $W_8$ . The right panel of figure 4 displays curves showing the effect of a ten percent change in  $W_8$  in gray. The impact of such a change in  $W_8$  quickly dissipates beyond the lowest eigenvalue, however, even a small 10% variation in  $W_8$  produces a clear effect on the spectrum, and we conclude that the value of  $W_8$  can be constrained to within at least ten percent using this procedure.

A complicating factor, however, is the potential presence of  $W_6$  and  $W_7$  in the effective theory. While the assumption of their suppression is not unreasonable, they may in fact provide a possible explanation for the observed lack of structure in the spectra. In our RMT setup they can straightforwardly be included in the fits and we refer to [9, 14] for details on the implementation in the Wilson RMT. We find (cf. figure 5) that the optimal fit to single eigenvalue distributions including now the effects from  $W_6, W_7$  and  $W_8$  indeed leads to an absence of structure in the aggregate spectrum. At this point, however, the effects of the various LECs are hard to disentangle and the uncertainties



**Figure 5:** Fits to the low lying eigenvalues of the Wilson Dirac operator in the charge sector  $v = 1$ , including the effects of all three LECs  $W_{6,7,8}$ . Results are for a volume of  $24^3 \times 24$  at  $\beta = 6.2$ .

on each of the parameters is considerably larger than when just taking  $W_8$  into account. Since the separate eigenvalues show a varying sensitivity to the different LECs, performing such fits at larger volumes, where more eigenvalues are available within the  $\varepsilon$ -regime, may help pinning down the various predictions.

#### 4. Conclusions

Recent developments which include lattice artefacts into the RMT allow for detailed predictions of the low lying eigenvalue spectrum of the Wilson Dirac operator in the  $\varepsilon$ -regime of QCD and provides a method for determining low energy constants of Wilson chiral perturbation theory. In these proceedings, we have reported on a preliminary study concerning the practical applicability of these predictions and we examined the sensitivity to the LECs. A potentially important source of uncertainty for these fits lies in the correct separation of configurations into topological sectors. We have studied the use of a field theoretical definition of the topological charge as a predictor for the overlap index which is the natural choice in connection with the Wilson Dirac operator spectrum. A high degree of correlation between the two definitions is found, although the field theoretical approach becomes unreliable as the volume is increased. If the LECs  $W_6$  and  $W_7$  are assumed to be zero, we find that  $W_8$  can be constrained to within about ten percent. For this purpose, fits to the separate eigenvalues, currently only available from direct RMT Monte Carlo calculations, appear to be the most powerful tool. The particular pattern of deviations seen in the separate eigenvalues, however, seems to point to a non-negligible contribution from the additional LECs  $W_6$  and  $W_7$ . For a more elaborate and more quantitative analysis along the lines described here (and consistently using the topological charge as determined from the eigenvalue flow), we refer to our detailed results presented in [14]. Furthermore, we would also like to point to the work by Heller et al. [15] who presented the results of a similar study at this conference [16].

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